Discrete Mathematics 50 (1984) 135–141
North-Holland

AN ASYMPTOTIC EXPANSION FOR THE COEFFICIENTS OF SOME POWER SERIES II: LAGRANGE INVERSION

Edward A. BENDER
University of California, San Diego, La Jolla, CA 92093, USA

L. Bruce RICHMOND*
University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

Received 8 July 1982
Revised 8 September 1982

Let $A(x)$ be a formal power series with rapidly growing coefficients and let $F(x)$ be analytic and zero at $x=0$. We develop asymptotic expansions for the coefficients of $A^{-1}(x)$ and $A(F(x))$. Applications are made to graphs, permutations, and set partitions.

1. Introduction

In [2] one of us proved a theorem which makes it a simple matter to determine asymptotic expansions for analytic functions of a variety of formal power series. Here we establish two related theorems. The first has applications to Lagrange inversion. The second deals with a formal power series of an analytic function.

Theorem 1. *Let $A(x)$ be a formal power series with coefficients a_n where $a_0=0$ and $a_1\neq 0$. Let p_n be the coefficient of x^n in $(1+A(x))^{\alpha n+\beta}$ where $\alpha\neq 0$ and β are fixed complex numbers. Let $R>0$ be a fixed integer.*

(i) *If $na_{n-1}\sim\gamma a_n$, then*

$$p_n=\alpha e^{\alpha a_1\gamma}na_n+O(a_n).$$

(ii) *If $na_{n-1}=o(a_n)$, then*

$$p_n=\sum_{k=0}^{R-1} d_{n,k}a_{n-k}+O(n^{R+1}a_{n-R}),$$

where $D_n(x)=\sum d_{n,k}x^k=(\alpha n+\beta)(1+A(x))^{\alpha n+\beta-1}$.

(iii) *In either case, $p_{n-1}=o(p_n)$ and*

$$\sum_{k=R}^{n-R}|p_kp_{n-k}|=O(p_{n-R}).$$

* Research supported by the NSERC under grant A4067.

Theorem 2. *Let $A(x)$ be a formal power series with coefficients a_n satisfying $na_{n-1}=o(a_n)$. Let $F(x)$ be a power series with non-zero radius of convergence, $f_0=0$ and $f_1\neq 0$. Fix $R>0$. The coefficient of x^n in $A(F(x))$ is*

$$\sum_{k=0}^{R-1} e_k a_{n-k} + \mathcal{O}(f_1^n n^R a_{n-R}),$$

where e_k is the coefficient of x^n in $F(x)^{n-k}$.

2. Applications

Let $[x^n]f(x)$ denote the coefficient of x^n in $f(x)$. The Lagrange inversion formula states

$$[x^n]g(h^{-1}(x)) = \frac{1}{n}[x^{n-1}]g'(x)(x/h(x))^n.$$

We give some examples of using this with Theorem 1 to obtain asymptotics.

Example 1. *Blocks of graphs.* Let $G(x)$, $C(x)$ and $B(x)$ be the exponential generating functions for graphs, connected graphs and blocks respectively. (All graphs are labelled.) It is known [4, (1.2.6) and (1.3.3)] that

$$C(x)=\log(1+G(x)), \qquad \log C'(x)=B'(xC'(x)). \tag{2.1}$$

Since $g_n=2^{n(n-1)/2}$, (2.1a) can be used to determine c_n:

$$c_n = \sum_{k=0}^{R-1} f_k g_{n-k} + \mathcal{O}(g_{n-R}), \tag{2.2}$$

where $F(x)=(1+G(x))^{-1}$. (See [8, Theorem 2] or [2, Corollary 4].) Using Lagrange inversion on (2.1b) with $h(x)=xC'(x)$ and $g(u)=\log C'(u)$ gives

$$\frac{b_{n+1}}{n!}=\frac{1}{n}[x^{n-1}]\left(\frac{C''(x)}{C'(x)}\left(\frac{1}{C'(x)}\right)^n\right),$$

so

$$b_{n+1}=(n-1)!\,[x^{n-1}]\left(-\frac{1}{n}(C'(x))^{-n}\right)' = -(n-1)!\,[x^n](C'(x))^{-n}.$$

Since $c_n \sim g_n = 2^{n(n-1)/2}$, Theorem 1(ii) with $A=C'-1$, $\alpha=-1$, $\beta=0$, gives a formula for the number of biconnected graphs. The main calculations are

$$F(x)=1-x-\tfrac{1}{3}x^3-\cdots, \qquad A(x)=C'(x)-1=x+2x^2+\cdots,$$

$$D_n(x)=-n+(n^2+n)(x-(n-4)x^2/2+\cdots),$$

$$c_n=g_n-ng_{n-1}+\mathcal{O}(n^3 g_{n-3}), \quad \text{where } g_n=2^{n(n-1)/2},$$

$$\begin{aligned} p_n &= -\frac{1}{(n-1)!}(c_{n+1}-(n^2+n)c_n+(n^3-n)(n-4)c_{n-1}/2+\mathcal{O}(n^6 c_{n-2}))\\ &= -\frac{1}{(n-1)!}(g_{n+1}-(n+1)^2 g_n+n(n+1)(n-2)^2 g_{n-1}/2+\mathcal{O}(n^6 g_{n-2})). \end{aligned} \tag{2.3}$$

Thus

$$b_n = g_n - n^2 g_{n-1} + n(n-1)(n-3)^2 g_{n-2}/2 + \mathcal{O}(n^6 g_{n-3}), \tag{2.4}$$

where $g_n = 2^{n(n-1)/2}$.

This gives the well-known result that almost all graphs are biconnected and furthermore gives an asymptotic estimate of the graphs which are not biconnected, namely $n^2 g_{n-1}$.

This idea can also be used for bipartite blocks. In this case (2.1a) becomes $C(x) = \frac{1}{2}\log(1+M(x))$, where $M(x)$ is the exponential generating function for $M_n = \sum \binom{n}{i} 2^{i(n-i)}$, the number of bicolored graphs [5]. In this case

$$\begin{aligned} b_n &= c_n - n(n-1)c_{n-1} + \mathrm{O}(n^4 c_{n-2}) \\ &= \tfrac{1}{2}M_n - \frac{(n^2-n+2)}{2} M_{n-1} + \mathrm{O}(M^4 M_{n-2}). \end{aligned} \tag{2.5}$$

The leading term was obtained by Wright [9].

Bessinger [1, Sec. 4.3] gives a variety of applications of the equation

$$A(x) = 1 + I(xA(x)), \tag{2.6}$$

to 'noncrossing compositions'. Here A and I are ordinary generating functions. She uses Lagrange inversion to obtain

$$i_n = -\frac{1}{n-1}[x^n]a(x)^{1-n}. \tag{2.7}$$

We consider asymptotics for some of her applications.

Example 2. *Permutations without invariant subintervals.* Let i_n be the number of permutations π on $[1, n] = \{1, \ldots, n\}$ with no invariant proper subintervals $[i, j]$. (S is invariant if $\pi(S) = S$.) Then [1, p. 90] (2.5) holds with $a_n = n!$. Theorem 1(i) can be used for (2.7) with $\alpha = -1$, $\beta = 1$, to obtain $i_n = n!/\mathrm{e} + \mathcal{O}((n-1)!)$. In other words, almost every derangement has no invariant subintervals, as one might expect.

Example 3. *Irreducible partitions.* A partition of $[1, n] = \{1, \ldots, n\}$ is called irreducible if no proper subinterval of $[1, n]$ is a union of blocks. If i_n is the number of irreducible partitions of $[1, n]$ and a_n is the total number where the block sizes are drawn from some specified set $\mathcal{B}$, then (2.6) holds. [1, Section 4.3a].

If $\mathcal{B}$ consists of all possible sizes then $a_n = B_n$, the Bell numbers, and $na_{n-1}/a_n = \log s + o(1)$, where $s \log s = n$. Thus a_n grows too slowly to apply Theorem 1; however, Theorem 1(i) suggests that $a_n/i_n \sim \mathrm{e}^{\log s} \sim n/\log n$. This is indeed correct, but entirely different methods are needed [3].

When $\mathcal{B} = \{k\}$, a_n counts k-diagrams and i_n counts linked k-diagrams. In this case

$$a_n = \begin{cases} n!/(k!)^{n/k}(n/k)! & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

We cannot apply Theorem 1 directly to $A(x)$ because many a_n are zero. Define $B(x)=A(x^{1/k})$ and consider the equation $B(x)=1+P(xB(x)^k)$. It is easily seen that $p_n=i_{nk}$. With $xB(x)^k=h(x)$ and $g(u)=B(u)$,

$$i_{nk}=p_n=\frac{1}{n}[x^{n-1}]B'(x)B(x)^{-nk}=\frac{1}{nk-1}[x^n]B(x)^{1-nk}.$$

We have $b_n/b_{n-1}\sim(nk)^{k-1}/(k-1)!$, so Theorem 1(i) applies when $k=2$ to give $i_n\sim a_n/\mathrm{e}$, a result obtained by Kleitman [6]. When $k>2$, Theorem 1(ii) applies to give

$$i_{nk}=\sum_{i=0}^{R-1}e_{n,i}\frac{(k(n-i))!}{k!^{n-i}(n-i)!}+\mathcal{O}\left(n^R\frac{(k(n-R))!}{k!^{n-R}(n-R)!}\right),$$

where

$$\sum_{i=0}^{\infty}e_{n,i}x^i=\left(1+\sum_{i=1}^{\infty}\frac{(ki)!}{k!^i i!}x^i\right)^{-nk}$$
$$=1-nkx+\tfrac{1}{2}nk\left(nk+1-\binom{2k}{k}\right)x^2+\cdots.$$

Naturally, entirely different methods are needed for Lagrange inversion when the power series converges. For example, the coefficient of x^{n-4} in

$$(1-2x)^{2n-1}(1+2x)/(1-x-x^2)^{3n+1}$$

can be estimated using the saddle point method [7].

We now apply Theorem 2.

Example 4. *Communication networks.* Suppose we wish to divide n people into groups and set up links between groups so that there is a communication path from every group to every other group. If $K(x)$ is the exponential generating function for the numbers, one easily has $K(x)=C(\mathrm{e}^x-1)$ where C is the exponential generating function for connected graphs. Theorem 1 is applicable.

$$k_n=\sum_{k=0}^{R-1}S(n,n-k)c_{n-k}+\mathcal{O}(n^{2R}c_{n-R}),$$

which can also be proved directly. By (2.3)

$$k_n=g_n-\tfrac{1}{2}n(n-3)g_{n-1}+\tfrac{1}{24}n(n-1)(3n^2-23n+22)g_{n-2}+\mathcal{O}(n^6g_{n-3}).$$

One can vary the problem by requiring two distinct paths between every pair of points. Then $B(x)$ replaces $C(x)$ and (2.4) replaces (2.3). If we divide $3n$ people into groups such that each group has 3 equal shifts, then e^x-1 is replaced by $\sum((3k)!/(k!)^3)x^k/k!$, the parenthesized term counting the number of ways to split $3k$ people into 3 equal shifts.

Example 5. *Smooth Graphs.* Wright [10] has defined a smooth labelled graph to be a connected graph without loops, multiple edges, or vertices of degree one. He obtains the exponeitial generating function $C(xe^{-x})-x+\frac{1}{2}x^2$ where $C(x)$ is given by (2.1a). In this case Theorem 2 applies with $A(x)=C(x)$ and

$$e_k=[x^n](xe^{-x})^{n-k}=[x^k]e^{-(n-k)x}=\frac{(-1)^k(n-k)^k}{k!}$$

and so the number of smooth graphs on n vertices is

$$\sum_{k=0}^{R-1}(-1)^k(n-k)^k\binom{n}{k}c_{n-k}+\mathcal{O}(n^{2R}c_{n-R}), \tag{2.8}$$

from which we obtain the expected result that the number of non-smooth connected graphs is asymptotic to n^2c_{n-1}. We can say more: Interpreting (2.8) as an inclusion–exclusion result, we see that a typical graph which can be made smooth by the deletion of k points is a connected graph with vertices of degree one attached.

3. Proofs

Let $N=\{1,2,\ldots,\}$ and $|\boldsymbol{k}|=k_1+k_2+\cdots$. We let $[m,n]$ denote a sum over all $\boldsymbol{k}\in N^m$ such that $|\boldsymbol{k}|=n$. A prime denotes the sum restricted to $k_1\leqslant N-s$ for all i. Define $d_{\boldsymbol{k}}=d_{k_1}d_{k_2}\cdots$.

Lemma 1. *Suppose $d_n\neq 0$ for $n>0$, $d_{n-1}=o(d_n)$, and for some $R\geqslant 2$*

$$\sum_{k=R}^{n-R}d_k\,d_{n-k}=\mathcal{O}(d_{n-R}). \tag{3.1}$$

Then for some constant C independent of m

$$\sum_{[m,n]}|d_{\boldsymbol{k}}|\leqslant C^{m-1}|d_{n-m+1}|, \tag{3.2}$$

$$\sum_{[m,n]}{}'|d_{\boldsymbol{k}}|\leqslant C^{m-1}|d_{n-s}|\quad \text{for } 0\leqslant s\leqslant R, \tag{3.3}$$

$$\left|\sum_{[m,n]}d_{\boldsymbol{k}}-md_1^{m-1}d_{n-m+1}\right|\leqslant C^m|d_{n-m}|. \tag{3.4}$$

Proof. Equations (3.2) and (3.3) are simply Lemma 2 of [2]. To prove (3.4), consider a term in the sum for which at least two $k_i\neq 1$. Note that if $\boldsymbol{k}\in N^{m-1}$ has some $k_i\neq 1$, then $|\boldsymbol{k}|\geqslant m$. Thus

$$\left|\sum_{[m,n]}d_{\boldsymbol{k}}-md_1^{m-1}d_{n-m+1}\right|\leqslant m\sum_{i=2}^{n-m}\left|d_i\right|\left|\sum_{[m-1,n-i]}\right|d_{\boldsymbol{k}}\bigg|.$$

By (3.2), the inner sum is at most $|C^{m-2}d_{n-i-m+2}|$. Apply (3.1), adjusting C if necessary. $\square$

Proof of Theorem 1. It is easily seen that $nd_{n-1}=\mathcal{O}(d_n)$ gives (3.1) for every R. Note that $d_n\neq 0$ can be relaxed to $d_n\neq 0$ for sufficiently large n to get (3.2) and (3.3) for sufficiently large n and $n-m$. We have

$$p_n=\sum_{m=1}^{n}\binom{\alpha n+\beta}{m}\sum_{[m,n]}a_k. \tag{3.5}$$

Suppose $na_{n-1}\sim\gamma a_n$. Then for every fixed m, $\binom{\alpha n+\beta}{m}\sim(\alpha n)^m/m!$, and by (3.4)

$$\sum_{[m,n]}a_k=md_1^{m-1}a_{n-m+1}+\mathcal{O}(a_{n-m})\sim ma_1^{m-1}a_n(\gamma/n)^{m-1}.$$

Thus, for some slowly growing function $\omega(n)$

$$\sum_{m=1}^{\omega(n)}\binom{\alpha n+\beta}{m}\sum_{[m,n]}a_k\sim\sum_{m}\frac{(\alpha m)^m}{m!}ma_1^{m-1}a_n(\gamma/n)^{m-1}\sim\alpha ne^{\alpha a_1\gamma}a_n.$$

The tail of (3.5) is negligible.

Now suppose $na_{n-1}=o(a_n)$. By Lemma 1 we have

$$\begin{aligned}\sum_{[m,n]}a_k&=m\sum_{j=m-1}^{R-1}a_{n-j}\sum_{[m-1,j]}a_k+\sum_{[m,n]}{}'a_k\\&=m\sum_{j=m-1}^{R-1}a_{n-j}\sum_{[m-1,j]}a_k+\mathcal{O}(a_{n-R})\end{aligned}$$

for $m\leqslant R$ and for $m>R$

$$\sum_{[m,n]}a_k\sim\mathcal{O}(C^{m-1}a_{n-m+1}).$$

Thus (3.5) becomes

$$\begin{aligned}p_n=&\sum_{j=0}^{R-1}\left(\sum_{m=1}^{j+1}m\binom{\alpha n+\beta}{m}\sum_{[m-1,j]}a_k\right)a_{n-j}+\mathcal{O}(n^Ra_{n-R})\\&+\sum_{m>R}\binom{\alpha n+\beta}{m}\mathcal{O}(C^{m-1}a_{n-m+1}).\end{aligned}$$

The parenthesized sum on m is $d_{n,j}$. Using $na_{n-1}=o(a_n)$, the final sum is

$$\begin{aligned}&\mathcal{O}\left(\sum_{m>R}\frac{(\alpha n)^m}{m!}C^mn^{R+1-m}a_{n-R}\right)=\\&=\mathcal{O}\left(\sum_{j\geqslant 0}\frac{(\alpha n)^{R+j+1}}{j!}C^jn^{-j}a_{n-R}\right)=\mathcal{O}(n^{R+1}a_{n-R}),\end{aligned}$$

which completes the proof of Theorem 1(ii).

Part (iii) follows easily from

$$\frac{np_{n-1}}{p_n}\sim\frac{na_{n-1}}{a_n}=\mathcal{O}(1).\qquad\square$$

We now prove Theorem 2, beginning with

Lemma 2. *Let $F(x)=\sum f_k x^k$ have radius of convergence greater than r. Suppose $f_0=0$, $f_1\neq 0$, and $f_i=0$ for $1<i<l$. Set $P(x)=\sum |f_k|\, x^k$. For $t\leqslant n(l-1)/l$ and $u=[t/(l-1)]$*

$$|[x^n]F(x)^{n-t}|\leqslant \binom{n-t}{u}|f_1|^n\, P(r)^u/|f_1 r|^{u+t}.$$

Proof. To get a nonzero term in the expansion of $F(x)\cdots F(x)$ we must use $f_1 x$ or $f_i x^i$ with $i\geqslant l$. If there are w of the latter, the power of x is at least $lw+(n-t-w)$. Since we wish x^n, $w\leqslant u$. Thus the term contains at least $n-t-u$ factors of $f_1 x$ and so

$$|[x^n]F(x)^{n-t}|\leqslant \binom{n-t}{n-t-u}|f_1|^{n-t-u}[x^{t+u}]P(x)^u.$$

Clearly $[x^{t+u}]P(x)^u\leqslant P(r)^u/r^{t+u}$. □

Proof of Theorem 2. Set $l=2$ in Lemma 2 and use $na_{n-1}=o(a_n)$ to obtain

$$\sum_{t\geqslant R} a_{n-t}[x^n]f(x)^{n-t}=$$

$$=\mathcal{O}\Big(f_1^n\sum_{t\geqslant R}\binom{n-t}{t}C^t\,|a_{n-t}|\Big)=\mathcal{O}\Big(f_1^n\sum_{t\geqslant R}n^R C^t a_{n-R}/t!\Big)=\mathcal{O}(f_1^n n^R a_{n-R}).$$

□

Note added in proof. A.M. Odlyzko and H. Wilf have provided (private communication) a quick direct proof that permutations without fixed points are almost all of the permutations without fixed intervals (see Example 2).

References

[1] J.S. Beissinger, Factorization and enumeration of labeled combinatorial objects, Ph.D. Thesis, University of Pennsylvania, (1981).

[2] E.A. Bender, An asymptotic expansion for the coefficients of some formal power series, J. London Math. Soc. (2) 9 (1975) 451–458.

[3] E.A. Bender and L.B. Richmond, The asymptotic number of irreducible partitions, Eur. J. Comb., to appear.

[4] F. Harary and E.M. Palmer, Graphical enumeration (Academic Press, New York, 1973).

[5] F. Harary and R.W. Robinson, Labeled bipartite graphs, Canad. J. Math. 31 (1979) 60–68.

[6] D.J. Kleitman, Proportions of irreducible diagrams, Stud. Appl. Math. 49 (1970) 297–299.

[7] R.C. Mullin, L.B. Richmond and R.G. Stanton, An asymptotic relation for bicubic maps, Proc. Third Manitoba Conf. on Numer. Math. (1973) 345–355.

[8] E.M. Wright, Asymptotic relations between enumerative functions in graph theory, Proc. London Math. Soc. 20 (1970) 558–572.

[9] E.M. Wright, The k-connectedness of bipartite graphs, J. London Math. Soc. (2) 25 (1982) 7–12.

[10] E.M. Wright, Enumeration of smooth labelled graphs, Proc. Royal Soc. Edinburgh (A) 91 (1982) 205–212.